Linear Transformations (L.T.)

Def: If $V(F)$ and $W(F)$ are two vector spaces, then a mapping T from V to W ie $T: V \to W$ is said to be linear transformation (or vector space homomorphism or linear mapping) iff

(i) $T(v+w) = T(v) + T(w) \;\forall\; v, w \in V$

(ii) $T(\alpha v) = \alpha T(v) \;\forall\; v \in V$ and $\alpha \in F$.

Or, If $V(F)$ and $W(F)$ are vector spaces Then $T: V \to W$ is L.T. iff $T(\alpha v + \beta w) = \alpha T(v) + \beta T(w)$ $\forall\, v, w \in V, \alpha, \beta \in F$.

2. Def. Linear operator: If $V(F)$ is a vector space. Then the linear transformation $T: V \to V$ is called linear operator (L.O.)

3. Def Linear Functional:- If $V(F)$ is a vector space, then the linear transformation $T: V \to F$ is called Linear functional.

1. Show that the following mapping are linear transformations:

(i) $T: V_3(R) \to V_2(R)$ defined by $T(x, y, z) = (x - y + z, 2x)$

(ii) $T: R^3 \to R^2$ defined by $T(x, y, z) = (z, x+y)$

Sol: (i) let $u = (x_1, y_1, z_1)$ and $v = (x_2, y_2, z_2) \in V_3(R)$ and α, β be two real

Given mapping is Linear Transformation if $T(\alpha u + \beta v) = \alpha T(u) + \beta T(v)$

Now $\alpha u + \beta v = \alpha(x_1, y_1, z_1) + \beta(x_2, y_2, z_2)$

$$= (\alpha x_1 + \beta x_2, \alpha y_1 + \beta y_2, \alpha z_1 + \beta z_2) \in V_3(R)$$

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Now $T(\alpha u+\beta v) = T(\alpha x_1+\beta x_2, \alpha y_1+\beta y_2, \alpha z_1+\beta z_2)$. let u

$= ((\alpha x_1+\beta x_2)-(\alpha y_1+\beta y_2)+(\alpha z_1+\beta z_2), 2(\alpha x_1+\beta x_2))$ Then

$= (\alpha x_1-\alpha y_1+\alpha z_1+\beta x_2-\beta y_2+\beta z_2, 2\alpha x_1+2\beta x_2)$ check

$= (\alpha(x_1-y_1+z_1)+\beta(x_2-y_2+z_2), 2\alpha x_1+2\beta x_2)$

$= (\alpha(x_1-y_1+z_1), 2\alpha x_1)+(\beta(x_2-y_2+z_2), 2\beta x_2)$

$= \alpha(x_1-y_1+z_1, 2x_1)+\beta(x_2-y_2+z_2, 2x_2)$

$= \alpha T(x_1, y_1, z_1)+\beta T(x_2, y_2, z_2)$

$= \alpha T(u)+\beta T(v)$.

Hence T is linear Transformation.

IInd part:- Similarly as part Ist.

(2). Show that the following mappings are not linear transformations

(i) $T: V_3(R) \to V_2(R)$ defined by $T(x, y, z) = (|y|, 0)$

(ii) $T: R^2 \to R^3$ defined by $T(x, y) = (x+1, 2y, x+y)$

(i) Sol: let $u=(x_1, y_1, z_1)$ and $v=(x_2, y_2, z_2) \in V_3(R)$

Then $u+v = (x_1+x_2, y_1+y_2, z_1+z_2) \in V_3(R)$

Now $T(u+v) = T(x_1+x_2, y_1+y_2, z_1+z_2)$

$= (|y_1+y_2|, 0)$ ——— (1)

and $T(u)+T(v) = T(x_1, y_1, z_1)+T(x_2, y_2, z_2) = (|y_1|, 0)+(|y_2|, 0)$

$= (|y_1|+|y_2|, 0)$ ——— (2)

From (1) and (2)

$T(u+v) \neq T(u)+T(v)$

Hence T is not a linear transformation.

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$+\frac{3}{2}2)$

$\frac{3}{2}2)$

. let $u = (x_1, y_1)$ and $v = (x_2, y_2) \in R^2$

Then $u+v = (x_1+x_2, y_1+y_2) \in R^2$

Check out $T(u+v) \neq T(u) + T(v)$

Hence T is not linear transformation.

(3). Find out which of the following mappings are linear transformations

(i) $T: R \to R^2$ defined as $T(x) = (2x, 3x)$ — (L.T).

(ii) $T: R^2 \to R$ defined by $T(x, y) = x - y$ — (L.T.)

(iii) $T: R^3 \to R^3$ defined by $T(x, y, z) = (x+2y, z)$ — (Not L.T.)

Zero Transformation (or operator) If $V(F)$ and $W(F)$ are vectorspaces then a mapping $T: V \to W$ defined as $T(x) = 0 \ \forall \ x \in W$ is a zero transformation.

Identity operator: If $V(F)$ is a vector space, then the mapping T defined as $T(x) = x \ \forall \ x \in V$ is a linear operator on V

Negative of a Linear Transformation: If $V(F)$ and $W(F)$ are vector space and $T: V \to W$ is a linear transformation, Then the linear Transformation mapping $-T: V \to W$ defined as $(-T)x = -[T(x)]$, is called negative of a linear transformation.

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Q: Find a L.T. $T: R^2 \to R^2$ such that $T(1,2) = (3,4)$ and $T(0,1)$

Sol: firstly we shall show that the given vectors of domain of T form a basis for R^2 (= domain of T).

To show: $(1,2)$, and $(0,1)$ are L.I

let $\alpha(1,2) + \beta(0,1) = 0$ for any sclar α, β.

$\Rightarrow (\alpha, 2\alpha+\beta) = (0,0)$

$\Rightarrow \alpha = 0, \quad 2\alpha+\beta = 0$
$\beta = 0.$

$\therefore (1,2)$ and $(0,1)$ are L.I.

To show $(1,2)$ and $(0,1)$ span R^2.

let $x, y \in R^2$

let $(x,y) = \alpha(1,2) + \beta(0,1)$

ie $(x,y) = (\alpha + 0\beta, 2\alpha+\beta)$

ie $\alpha = x, \quad y = 2\alpha + \beta$

$\beta = y - 2\alpha$
$= y - 2x$

$\therefore (x,y) = x(1,2) + (y-2x)(0,1)$

Hence $(1,2)$ $(0,1)$ span R^2.

$\therefore T(x,y) = T(x(1,2) + (y-2x)(0,1))$

$= x\,T(1,2) + (y-2x)\,T(0,1)$

$= x(3,4) + (y-2x)(0,0)$

$= (3x, 4x) + (0,0)$

$T(x,y) = (3x, 4x)$

which is required linear transformation.

Find $T(a,b,c)$ where $T: R^3 \to R$ is defined by

$T(1,1,1) = 3, \quad T(1,1,0) = -4, \quad T(1,0,0) = 2$

$T(1,1,1) = 3 \quad T(0,1,-2) = 1, \quad T(0,0,1) = -2.$

Ans – $T(x,y,z) = 8x - 3y - 2z$ (for 2nd Question).

Sol: firstly we shall show that the given vectors of domain of T form a basis for R^3 ($\doteq$ domain of T)

To show: $(1,1,1)$, $(1,1,0)$ and $(1,0,0)$ are L.I.

Consider $\alpha(1,1,1) + \beta(1,1,0) + \gamma(1,0,0) = 0$ for α, β, γ any scalar.

$\Rightarrow (\alpha+\beta+\gamma, \; \alpha+\beta, \; \alpha) = (0,0,0)$

$$\therefore \left.\begin{aligned} \alpha+\beta+\gamma &= 0 \\ \alpha+\beta &= 0 \\ \alpha &= 0 \end{aligned}\right] \longrightarrow \text{Solve that} \quad \begin{aligned} \alpha &= 0 \\ \beta &= 0 \\ \gamma &= 0 \end{aligned}$$

$\therefore (1,1,1)$, $(1,1,0)$ and $(1,0,0)$ are L.I.

Now To show $(1,1,1)$, $(1,1,0)$ and $(1,0,0)$ span R^3.

let $(x,y,z) \in R^3$

let $(x,y,z) = \alpha(1,1,1) + \beta(1,1,0) + \gamma(1,0,0)$

$= (\alpha+\beta+\gamma, \; \alpha+\beta, \; \alpha)$

$\therefore \alpha = z, \quad \alpha+\beta = y, \quad \alpha+\beta+\gamma = x$

$\Rightarrow \alpha = z, \quad \beta = y - z, \quad \gamma = x - y$

Thus $(x,y,z) = z(1,1,1) + (y-z)(1,1,0) + (x-y)(1,0,0)$

Hence $(1,1,1)$, $(1,1,0)$ and $(1,0,0)$ span R^3.

$\therefore T(x,y,z) = z\,T(1,1,1) + (y-z)\,T(1,1,0) + (x-y)\,T(1,0,0)$

$= z(3) + (y-z)(-4) + (x-y)(2)$

$= 2x - 6y + 7z$ which is required L.T.

IInd part – Similarly 1st part as.

Q:-3 let $v_1 = (1, 1, -1)$ $v_2 = (4, 1, 1)$ and $v_3 = (1, -1, 2)$ is a basis of R^3. let $T: R^3 \to R^2$ be L.T. Such that

$T(v_1) = (1,0)$, $T(v_2) = (0,1)$, $T(v_3) = (1,1)$ find T.

Sol:- let $v = (x, y, z) \in R^3$ be a arbitrary element

$\because$ $\{v_1, v_2, v_3\}$ is a basis of R^3.

So v can be expressed as L.C. of v_1, v_2, v_3

let $v = \alpha v_1 + \beta v_2 + \gamma v_3$ for some scalar α, β, γ

$\Rightarrow (x, y, z) = \alpha(1,1,-1) + \beta(4,1,1) + \gamma(1,-1,2)$

$= (\alpha + 4\beta + \gamma, \alpha + \beta - \gamma, -\alpha + \beta + 2\gamma)$

$\therefore$ $\left.\begin{array}{l} \alpha + 4\beta + \gamma = x \\ \alpha + \beta - \gamma = y \\ -\alpha + \beta + 2\gamma = z \end{array}\right]$ Solving $\begin{array}{l} \alpha = 3x - 7y - 5z \\ \beta = -x + 3y + 2z \\ \gamma = 2x - 5y - 3z. \end{array}$

So that $v = (3x - 7y - 5z)v_1 + (-x + 3y + 2z)v_2 + (2x - 5y - 3z)v_3$

$T(v) = (3x - 7y - 5z)T(v_1) + (-x + 3y + 2z)T(v_2) + (2x - 5y - 3z)T(v_3)$

$= (3x - 7y - 5z)(1,0) + (-x + 3y + 2z)(0,1) + (2x - 5y - 3z)(1,1)$

$= (5x - 12y - 8z, x - 2y - z)$

$\Rightarrow T(x, y, z) = (5x - 12y - 8z, x - 2y - z)$ is required L.T.

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find a linear transformation $T: R^2 \to R^2$ such that $T(1,0) = (1,1)$ and $T(0,1) = (-1,2)$.

Prove that T maps the square with vertices $(0,0), (1,0), (1,1)$, and $(0,1)$ into a parallelogram.

Sol. (Hint - $T(x,y) = (x-y, x+2y)$ is the required L.T.)

IInd part: let the vertices of square be P, Q, R, S resp. and their images be A, B, C, D resp.

$\therefore A = T(P) = T(0,0) = (0,0)$

$B = T(Q) = T(1,0) = (1,1)$

$C = T(R) = T(1,1) = (0,3)$

$D = T(S) = T(0,1) = (-1,2)$

mid point of $[AC] = \left(\frac{0+0}{2}, \frac{0+3}{2}\right) = \left(0, \frac{3}{2}\right)$

mid point of $[BD] = \left(\frac{1+(-1)}{2}, \frac{1+2}{2}\right) = \left(0, \frac{3}{2}\right)$

$\therefore$ mid point of $[AC]$ = mid point of $[BD]$

Thus $ABCD$ is a parallelogram.

Hence T maps square P, Q, R, S into a parallelogram $ABCD$.

Range, Rank and Nullity of Linear Transform

Range: If $V(F)$ and $W(F)$ are vector spaces and $T: V \to W$ is linear Transformation, then the image set of V under T is called the range of T, which is denoted by Range T or Image T or $R(T)$ or $T(V)$

i.e. $\text{Range } T = \{T(v) : v \in V\}$

Range T is also called Range Space. ($\because R(T)$ is vector space)

Null Space or Kernel: – If $V(F)$ and $W(F)$ are two vector spaces and $T: V \to W$ is a linear transformation then the set of all those vectors in V whose image under T is zero, is called Kernel or Null Space of T which is denoted by $N(T)$ i.e.

Null Space of $T = N(T) = \{v \in T(v) = 0 \in W\}$

Rank: If $V(F)$ and $W(F)$ be vector spaces and $T: V \to W$ be a L.T., then the dimension of range space of T is called the rank of T and is denoted by $\rho(T)$

Thus $\rho(T) = \dim(\text{Range } T)$

Nullity: – If $V(F)$ and $W(F)$ be vector spaces and $T: V \to W$ be a L.T. then the dimension of null space of T is called the nullity of T and is denoted by $\nu(T)$.

Thus $\nu(T) = \dim(\text{Null space of } T)$.

Rank-Nullity Theorem or (Sylvester's Law of nullity)

(without proof)

If $V(F)$ and $W(F)$ are vector spaces and $T: V \to W$ is a linear transformation. Suppose V is of dimension n (i.e. V is a finite dimensional) $\therefore$ Then. Rank T + Nullity T = dim V.

1. ~~For~~ each of following linear transformations $T: V \to W$.
find a basis and dimension of (one or two parts)
(i) its Range space
(ii) its null space
Also verify Rank (T) + Nullity (T) = dim V i.e. Rank Nullity Th.

(a) $T: R^2 \to R^3$ defined by $T(x,y) = (x+y,\ x-y,\ y)$

(b) $T: R^3 \to R^3$ defined by $T(x,y,z) = (x+2y,\ y-z,\ x+2z)$

(c) $T: R^3 \to R^3$ defined by $T(x,y,z) = (x+2y-z,\ y+z,\ x+y-2z)$

(d) $T: R^4 \to R^3$ be defined by
$T(x,y,z,t) = (x-y+z+t,\ x+2z-t,\ x+y+3z-3t)$

(e) $T: R^2 \to R^3$ be defined by
$T(x,y) = (x-y,\ y-x,\ -x)$

(f) $T: R^2 \to R^2$ be defined by
$T(x,y) = (x+y,\ x-y)$.

Sol:- We know that a basis for R^2 is
$B = \{e_1, e_2\} = \{(1,0), (0,1)\}$

(i) Firstly we shall find basis for Range T.
Since B is a basis of R^2
$\therefore B_1 = \{T(e_1), T(e_2)\}$ generates Range T.

Here $T(e_1) = T(1,0) = (1+0, 1-0, 0) = (1,1,0)$
$T(e_2) = T(0,1) = (0+1, 0-1, 1) = (1,-1,1)$

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$\therefore$ $B_1 = (1,1,0), (1,-1,1)$ generates range T. ——— (1)

To find basis of range T, we have to find L.I. vectors from $T(e_1)$, $T(e_2)$.

Consider $\alpha(1,1,0) + \beta(1,-1,1) = (0,0,0)$ for α, β any scalars.

$\Rightarrow (\alpha+\beta, \alpha-\beta, \beta) = (0,0,0)$

$\therefore \alpha+\beta = 0, \quad \alpha-\beta = 0, \quad \alpha = 0$

$\Rightarrow \alpha = 0, \ \beta = 0$

$\therefore (1,1,0), (1,-1,1)$ are L.I. vectors

$\Rightarrow B_1$ is L.I. set ——— (2)

From (1) and (2) B_1 is a basis of $R(T)$

$\Rightarrow$ Range space of $T = \{(1,1,0), (1,-1,1)\}$

$\therefore$ Rank (T) = Number of elements in $B_1 = 2$.

II. To find basis for Null Space of T.

Let $v = (x,y) \in N(T)$

$\Rightarrow T(v) = T(x,y) = 0$

$\Rightarrow (x+y, x-y, y) = (0,0,0)$

$\Rightarrow x+y = 0$

$x-y = 0$

$y = 0$

ie $x = 0, \ y = 0$

$\Rightarrow v = (0,0)$

So that $v \in N(T) \Rightarrow v = (0,0) = 0$

$\therefore$ Null space of $T = (0)$

and Nullity $T = \dim N(T) = 0$

Thus Nullity T + Rank $T = 0+2 = 2 = \dim R^2 = \dim V$

Hence the result.

We know $T: R$ ... $B = S$ ... (i)

$T: R^3 \to R^3$ defined $T(x,y,z) = (x+2y,\ y-z,\ x+2z)$

We know that a basis for R^3 is

$B = \{e_1, e_2, e_3\} = \{(1,0,0), (0,1,0), (0,0,1)\}$

(i) Firstly we shall find Basis for range T

$\because$ B is a basis of R^3

$\therefore B_1 = \{T(e_1), T(e_2), T(e_3)\}$ generates Range T

Here $T(e_1) = T(1,0,0) = (1,0,1)$

$T(e_2) = T(0,1,0) = (2,1,0)$

$T(e_3) = T(0,0,1) = (0,-1,2)$

$\left\{\because T(x,y,z) = (x+2y,\ y-z,\ x+2z)\right\}$

$\therefore B_1 = \{(1,0,1), (2,1,0), (0,-1,2)\}$ generates range T.

To find basis for range T, we have to find the L.I. vectors from $T(e_1), T(e_2), T(e_3)$. for this consider the matrix whose rows are generators of T and reduce it to echelon form

ie $A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \\ 0 & -1 & 2 \end{bmatrix}$

$R_2 \to R_2 - 2R_1$

$\sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & -1 & 2 \end{bmatrix}$

$R_3 \to R_3 + R_2$

$\sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$

$\therefore (1,0,1), (0,1,-2)$ form L.I set of vectors which generates range T.

So Range space of $T = \{(1,0,1), (0,1,-2)\}$

$\therefore$ Rank T = Number of elements in this basis = 2.

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To find basis for Null space

let $v \in (x, y, z) \in N(T)$

$\Rightarrow T(v) = T(x, y, z) = 0$

$\Rightarrow (x+2y,\ y-z,\ x+2z) = (0, 0, 0)$

$$\Rightarrow \left.\begin{array}{l} x+2y=0 \\ y-z=0 \\ x+2z=0 \end{array}\right] \text{———} (1)$$

To finding a basis of null space of T, it is equivalent to find a basis of the solution space of above equations

for this, matrix $P = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & 2 \end{bmatrix}$

$R_3 \to R_3 - R_1$

$$\sim \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & -1 \\ 0 & -2 & 2 \end{bmatrix}$$

$R_3 \to R_3 + 2R_2$

$$\sim \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

Another method

From (1)

$y = z$

$x = -2z$

$x + 2y = 0$

$x = -2y = -2z$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2z \\ z \\ z \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} z$$

$\therefore B_2 = \{(-2, 1, 1)\}$ is a basis for Null space of T.

$\therefore$ System of equations (1) is equivalent to

$x + 2y = 0 \Rightarrow x = -2y$

$y - z = 0 \Rightarrow y = z$

$\therefore$ Solution set is $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2y \\ y \\ y \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} y$

Hence $B_2 = \{(-2, 1, 1)\}$ is a basis for null space of T and $\dim(N(T)) = 1$.

$\Rightarrow$ Nullity $T = \dim(N(T)) = 1$.

$\therefore$ Nullity T + Rank $T = 1 + 2 = 3 = \dim R^3 = \dim V$. Proved.

Let $T: R^4 \to R^3$ be a linear transformation defined by
$T(e_1) = (1,1,1)$, $T(e_2) = (1,-1,1)$, $T(e_3) = (1,0,0)$, $T(e_4) = (1,0,1)$
find $R(T)$ and $N(T)$ and
verify Rank T + Nullity $T = 4 = \dim(R^4)$.

Sol: Given $T(e_1) = (1,1,1)$, $T(e_2) = (1,-1,1)$, $T(e_3) = (1,0,0)$
and $T(e_4) = (1,0,1)$

let $(x,y,z,t) \in R^4$

Then $(x,y,z,t) = x(1,0,0,0) + y(0,1,0,0) + z(0,0,1,0) + t(0,0,0,1)$
$= xe_1 + ye_2 + ze_3 + te_4$

$T(x,y,z,t) = xT(e_1) + yT(e_2) + zT(e_3) + tT(e_4)$
$= x(1,1,1) + y(1,-1,1) + z(1,0,0) + t(1,0,1)$
$= (x+y+z+t,\ x-y,\ x+y+t)$

To find $R(T)$:

we know $B = \{e_1, e_2, e_3, e_4\}$ is a basis of R^4

$\Rightarrow \{T(e_1), T(e_2), T(e_3), T(e_4)\}$ generates $R(T)$

$\Rightarrow \{(1,1,1), (1,-1,1), (1,0,0), (1,0,1)\}$ generates $R(T)$.

To find L.I. vectors from this set

Consider matrix A whose rows are generator of $R(T)$ and reduce it to echelon matrix

i.e. $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}$

$R_1 \leftrightarrow R_3$, $R_2 \leftrightarrow R_4$

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$R_2 \to R_2 - R_1$

$R_3 \to R_3 - R_1$

$R_4 \to R_4 - R_1$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix}$$

$R_4 \to R_4 + R_3$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix} \underset{R_4 \to R_4 - 2R_2}{\sim} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \underset{R_2 \leftrightarrow R_3}{\sim} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

which is echelon form

$\therefore$ $(1,0,0), (0,1,1)$ $(0,0,1)$ form L.I set of vectors which generates

Range $T = R(T)$

$\Rightarrow$ Range space of $T = \{(1,0,0), (0,1,1), (0,0,1)\}$

$\therefore$ Rank $T = \dim R(T) = 3$.

To find $N(T)$

let $(x,y,z,t) \in N(T) \subseteq R^4$

$\Rightarrow$ $T(x,y,z,t) = 0$

$\Rightarrow$ $(x+y+z+t,\ x-y,\ x+y+t) = (0,0,0)$

$\therefore$ $x+y+z+t = 0$

$x-y = 0$

$x+y+t = 0$

Solving $z = 0$, $x = y$, $t = -2x$

$\Rightarrow$ $(x,y,z,t) = (x, x, 0, -2x) = x(1,1,0,-2)$

$\therefore$ $N(T)$ is generated by $(1,1,0,-2)$

Null space of $T = \{(1,1,0,-2)\}$, Nullity $T = \dim(N(T)) = 1$.

Here Rank T + Nullity $T = 3+1 = 4 = \dim R^4$.

verify

sol:

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verify Rank - Nullity Theorem for matrix

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}_{3\times3}.$$

Sol: Dim V = number of columns of Matrix = 3

First part: To find Range space:

Take transpose of matrix A and convert to echelon form -

$$A^T = \begin{bmatrix} 2 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix}$$

$R_1 \to R_1 - 2R_3$, $R_2 \to R_2 - R_3$

$$\sim \begin{bmatrix} 0 & 0 & -3 \\ 0 & 1 & -1 \\ 1 & 0 & 2 \end{bmatrix}$$

$R_1 \leftrightarrow R_3$

$$\sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & -3 \end{bmatrix}$$

$(\frac{-1}{3})R_3$

$$\sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$\Rightarrow$ Range space $= \{(1,0,2), (0,1,-1), (0,0,1)\}$

is the basis set of Range A.

$\Rightarrow$ Rank A = dim (Range space) = 03.

IInd part: To find Null space:

let $v = (x, y, z) \in R^3$ be any vector in Null space

Then $AX = 0$

i.e. $\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Operate $R_1 \to R_1 - 2R_3$

Here $A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 0 & -1 & -3 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$

$R_1 \leftrightarrow R_3$

$\sim \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 0 \\ 0 & -1 & -3 \end{bmatrix}$

$R_3 \to R_3 + R_2$

$\sim \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & -3 \end{bmatrix}$

$-\frac{1}{3} R_3$

$\sim \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\therefore$ The system of equation reduces as

$x + y + 2z = 0$

$\boxed{y = 0}$

$\boxed{z = 0}$ $\quad \therefore \boxed{x = 0}$

ie $x = 0, \; y = 0, \; z = 0$

ie $v = (x, y, z) = (0, 0, 0)$

Null space basis set $= \{(0,0,0)\} = 0$

$\therefore$ Nullity $A = 0$

Hence Rank A + Nullity $A = 3 + 0 = 3 = \dim V$.

verified

verify

verify Rank Nullity Theorem for matrix

$$A = \begin{bmatrix} 2 & -1 & 3 & 1 \\ 1 & 4 & -2 & 1 \\ 5 & 2 & 4 & 3 \end{bmatrix}$$

Sol: Dim V = number of column of matrix = 04.

First part: To find Range space :

Take transpose of matrix A and Convert to echelon form

$$A^T = \begin{bmatrix} 2 & 1 & 5 \\ -1 & 4 & 2 \\ 3 & -2 & 4 \\ 1 & 1 & 3 \end{bmatrix}$$

$R_1 \leftrightarrow R_4$

$$\sim \begin{bmatrix} 1 & 1 & 3 \\ -1 & 4 & 2 \\ 3 & -2 & 4 \\ 2 & 1 & 5 \end{bmatrix}$$

$R_2 \to R_2 + R_1$, $R_3 \to R_3 - 3R_1$, $R_4 \to R_4 - 2R_1$

$$\sim \begin{bmatrix} 1 & 1 & 3 \\ 0 & 5 & 5 \\ 0 & -5 & -5 \\ 0 & -1 & -1 \end{bmatrix}$$

$R_2 \leftrightarrow R_4$

$$\sim \begin{bmatrix} 1 & 1 & 3 \\ 0 & -1 & -1 \\ 0 & -5 & -5 \\ 0 & 5 & 5 \end{bmatrix}$$

$R_3 \to R_3 - 5R_2$, $R_4 \to R_4 + 5R_2$

$$\sim \begin{bmatrix} 1 & 1 & 3 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad R_1 \to R_1 + R_2 \quad \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\longleftrightarrow R_2$

$$\sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\Rightarrow$ Range space $= \{(1,0,2), (0,1,1)\}$ is the basis of set of Range A.

$\Rightarrow$ Rank A = dim(Range space) = 02.

IInd part: <u>To find Null space:</u>

let $v \in (v_1, v_2, v_3, v_4) \in R^4$ be any vector in Null space, Then

$$AV = 0$$

ie
$$\begin{bmatrix} 2 & -1 & 3 & 1 \\ 1 & 4 & -2 & 1 \\ 5 & 2 & 4 & 3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{——— (1)}$$

Here $A = \begin{bmatrix} 2 & -1 & 3 & 1 \\ 1 & 4 & -2 & 1 \\ 5 & 2 & 4 & 3 \end{bmatrix}$ operate $R_1 \to R_1 - 2R_2$
$R_3 \to R_3 - 5R_2$

$$\sim \begin{bmatrix} 0 & -9 & 7 & -1 \\ 1 & 4 & -2 & 1 \\ 0 & -18 & 14 & -2 \end{bmatrix}$$

$R_3 \to R_3 - 2R_1$

$$\sim \begin{bmatrix} 0 & -9 & 7 & -1 \\ 1 & 4 & -2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ system of Equation (1) reduces as.

$-9v_2 + 7v_3 - v_4 = 0 \quad \Rightarrow v_4 = -9v_2 + 7v_3$

and $v_1 + 4v_2 - 2v_3 + v_4 = 0 \Rightarrow v_1 = -4v_2 + 2v_3 - v_4$

$= -4v_2 + 2v_3 - (-9v_2 + 7v_3)$

$= 5v_2 - 5v_3$

$$\therefore v = \begin{bmatrix} v_4 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ 0 \\ -9 \end{bmatrix} v_2 + \begin{bmatrix} -5 \\ 0 \\ 1 \\ 7 \end{bmatrix} v_3$$

$\therefore$ Null space basis set $= \{(5,1,0,-9), (-5,0,1,7)\}$

$\therefore$ Nullity A = dim (Null space) = 02

Hence Rank A + Nullity A = 02 + 02 = 04 = dim V (coloum number of matrix) (verified.)

find ... Trans... dim... (... st. ...

(40)

(4) Find a linear transformation $T: R^3 \to R^4$ whose range spanned by $(1, 2, 0, -4)$ and $(2, 0, -1, -3)$ Find

Sol: The usual basis of R^3 is

$$B = \{e_1, e_2, e_3\} = \{(1,0,0), (0,1,0), (0,0,1)\}$$

$\therefore$ The Range of T is generated by

$$B_1 = \{T(e_1), T(e_2), T(e_3)\}$$

But it is given that range is generated by $(1,2,0,-4)$ and $(2,0,-1,-3)$

let $T(e_1) = (1, 2, 0, -4)$

$T(e_2) = (2, 0, -1, -3)$

and $T(e_3) = (0, 0, 0, 0)$

for each $(x, y, z) \in R^3$, we have

$$(x, y, z) = x(1,0,0) + y(0,1,0) + z(0,0,1)$$

$$= xe_1 + ye_2 + ze_3$$

$$\therefore T(x,y,z) = T(xe_1 + ye_2 + ze_3)$$

$$= xT(e_1) + yT(e_2) + zT(e_3)$$

$$= x(1,2,0,-4) + y(2,0,-1,-3), + z(0,0,0,0)$$

$$= (x+2y, 2x-y, -4x-3y)$$

which is the required linear Transformation

ii) find a linear Transformation $T: P_3(x) \to P_2(x)$ such that
$T(1+x) = 1+x$, $T(2+x) = x+3x^2$, $T(x^2) = 0$

Sol: Firstly we show that $B = \{1+x,\ 2+x,\ x^2\}$ $(T: P_3(x) \to P_2(x))$
forms a basis of $P_3(x)$

(i) To Prove B is L.I –

let K_1, K_2, K_3 are 3 scalars such that
$K_1(1+x) + K_2(2+x) + K_3x^2 = 0$
$\Rightarrow (K_1+2K_2) + (K_1+K_2)x + K_3x^2 = 0 + 0x + 0x^2$

Equating like powers of x on both sides, we get

$K_1 + 2K_2 = 0$ —— (1)
$K_1 + K_2 = 0$ —— (2)
$\boxed{K_3 = 0}$ from (1) and (2), $K_1 = 0$, $K_2 = 0$

(ii) <u>To Prove B Spans $P_3(x)$</u>

let $a_0 + a_1x + a_2x^2 + a_3x^3 \in P_3(x)$
Then $a_0 + a_1x + a_2x^2 + a_3x^3 = \alpha_1(1+x) + \alpha_2(2+x) + \alpha_3 x^2$

$a_0 = \alpha_1 + 2\alpha_2$ —— (1)
$a_1 = \alpha_1 + \alpha_2$ —— (2) from (1) and (2)
$a_3 = 0$ —— (3) $\alpha_2 = a_0 - a_1$
$\boxed{a_2 = \alpha_3}$

Now, α_2 use in (2), $\Rightarrow \alpha_1 = 2a_1 - a_0$

Thus $a_0 + a_1x + a_2x^2 + a_3x^3 = (2a_1 - a_0)(1+x) + (a_0 - a_1)(2+x) + a_2(x^2)$
Thus B spans $P_3(x)$

Now $T(a_0 + a_1x + a_2x^2 + a_3x^3) = (2a_1 - a_0)T(1+x) + (a_0 - a_1)T(2+x) + a_2T(x^2)$
$= (2a_1 - a_0)(1+x) + (a_0 - a_1)(x+3x^2) + a_2(0)$
$= (-a_0 + 2a_1) + a_1x + 3(a_0 - a_1)x^2$ which is reqd. LT

Find a linear Transformation $T: R^3 \to R^3$

(a) whose image is generated by $(1,0,-1)$ and $(1,2,2)$

(b) whose range space is generated by $(1,2,3)$ and $(4,5,6)$

Sol. The usual basis of R^3 is $B = \{e_1, e_2, e_3\} = \{(1,0,0), (0,1,0), (0,0,1)\}$

$\therefore$ the range of T is generated by

$B_1 = \{T(e_1), T(e_2), T(e_3)\}$

But it is given that range (Image) is generated by $(1,0,-1)$ and $(1,2,2)$

let $T(e_1) = (1,0,-1)$, $T(e_2) = (1,2,2)$ and $T(e_3) = (0,0,0)$

for each $(x,y,z) \in R^3$ we have

$(x,y,z) = x(1,0,0) + y(0,1,0) + z(0,0,1)$

$= xe_1 + ye_2 + ze_3$

$T(x,y,z) = xT(e_1) + yT(e_2) + zT(e_3)$

$= x(1,0,-1) + y(1,2,2) + z(0,0,0)$

$= (x+y, 2y, -x+2y)$ which is required L.T.

(b) Home work.

(B) Find a linear map $T: R^4 \to R^3$ whose null space is generated by $(1,2,3,4)$ and $(0,1,1,1)$.

Sol.: let $N(T)$ be the null space of T since $N(T)$ is generated by $B = \{(1,2,3,4), (0,1,1,1)\} = \{v_1, v_2\}$ say

Here v_1 is not a scalar multiple of v_2

$\therefore$ B is L.I set and $\dim N(T) = 2$

We shall extend it to a Basis of R^4

Consider $B_1 = \{(1,0,0,0), (0,1,0,0), (0,0,1,0), (0,0,0,1)\}$

$\therefore B \cup B_1 = \{(1,2,3,4), (0,1,1,1), (1,0,0,0), (0,1,0,0), (0,0,1,0), (0,0,0,1)\}$

To find basis of $\mathbb{R}^4$, we have to find L.I vectors out elements of $B \cup B_1$. for this consider a matrix A,

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 1 & 1 \\ 0 & -2 & -3 & -4 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & \\ 0 & 1 & 1 & \\ 0 & 0 & -1 & -2 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_3 \to R_3 - R_1$
$R_4 \to R_4 - R_2$

$R_3 \to R_3 + 2R_2$
$R_5 \to R_5 + R_4$

$R_4 \to R_4 - R_3$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_5 \to R_5 + R_4$
$R_6 \to R_6 - R_4$

$R_3 \leftrightarrow (-1)R_3$

So that $B_2 = \{(1,2,3,4), (0,1,1,1)\ (0,0,1,2)\ (0,0,0,1)\}$ is a basis of $\mathbb{R}^4$ which is an extension of B.

Now we define a function $S: B_2 \to \mathbb{R}^3$ as

$$S(1,2,3,4) = (0,0,0)$$
$$S(0,1,1,1) = (0,0,0)$$
$$S(0,0,1,2) = (1,0,0)$$
$$S(0,0,0,1) = (0,1,0)$$

Extending the map S to linear transformation $T: \mathbb{R}^4 \to \mathbb{R}^3$

let $(x,y,z,t) = a(1,2,3,4) + b(0,1,1,1) + c(0,0,1,2) + d(0,0,0,1)$

$$= (a,\ 2a+b,\ 3a+b+c,\ 4a+b+2c+d)$$

Solve as, $a = x$, $b = y-2x$, $c = z-x-y$, $d = t+y-2z$

$\therefore (x,y,z,t) = x(1,2,3,4) + (y-2x)(0,1,1,1) + (z-x-y)(0,0,1,2) + (t+y-2z)(0,0,0,1)$

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$$\therefore T(x,y,z,t) = T\Big[x(1,2,3,4) + (y-2x)(0,1,1,1) + (z-x-y)(0,0,1,2) + (t+y-2z)(0,0,0,1)\Big]$$

$$= x(0,0,0) + (y-2x)(0,0,0) + (z-x-y)(1,0,0) + (t+y-2z)(0,1,0).$$

$$= (z-x-y,\ t+y-2z,\ 0)$$

$\therefore T(x,y,z,t) = (z-x-y,\ t+y-2z,\ 0)$ is the required linear Transformation

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(44)

1. **One-one (Injective) Transformation:-** let $T: V \to W$ be a linear transformation. Then T is called one-one if for all x, y

$$x \neq y \Rightarrow T(x) \neq T(y) \quad \text{or}$$

$$T(x) = T(y) \Rightarrow x = y.$$

2. **onto (Surjective) Transformation:-** let $T: V \to W$ be a linear transformation. Then T is called onto or surjective iff for each $w \in W$ $\exists$ $v \in V$ s.t. $w = T(v)$. or $W =$ Range of T.

3. **One-one, onto Transformation:-** let $T: V \to W$ be a L.T. Then T is called bijective iff it is both one-one (Injective) and onto (surjective).

4. **Non-singular Transformation:** A Linear Transformation $T: V \to W$ is said to be non-singular iff the null space of T is zero space $\{0\}$ ie The null space consists of only the zero element

Thus if $T(v) = 0 \Rightarrow v = 0$ for all $v \in V$

or if $v \neq 0 \Rightarrow T(v) \neq 0$ for all $v \in V$

then T is said to be non-singular.

5. **Singular Transformation:-** A linear transformation $T: V \to W$ is said to be singular iff the null space of T contains at least one non-zero vectors.

Thus if $v \neq 0 \Rightarrow T(v) = 0$ for some $v \in V$

Thus T is said to be singular

Let $T_1: R^3 \to R^2$ and $T_2: R^3 \to R^2$ be defined by
$T_1(x,y,z) = (3x+y, z)$ and $T_2(x,y,z) = (-y+z, x-y)$.
find T_1+T_2, $4T_1$, $3T_1-T_2$, T_1T_2, T_2T_1 if possible.

Sol:- Here (i) $T_1+T_2: R^3 \to R^2$ is defined by

$$(T_1+T_2)(x,y,z) = T_1(x,y,z) + T_2(x,y,z)$$
$$= (3x+y, z) + (-y+z, x-y)$$
$$= (3x+z, x-y+z).$$

(ii) $4T_1: R^3 \to R^2$ defined by

$$(4T_1)(x,y,z) = 4.T_1(x,y,z)$$
$$= 4(3x+y, z)$$
$$= (12x+4y, 4z)$$

(iii) $(3T_1-T_2) R^3 \to R^2$ defined by

$$(3T_1-T_2)(x,y,z) = 3T_1(x,y,z) - T_2(x,y,z)$$
$$= (9x+4y-z, -x+y+3z)$$

(iv) T_1T_2 is not defined since Range of T_1 $(=R^2)$ is not subset of domain of T_1 $(=R^3)$

(v) T_2T_1 is also not defined same case =

(2) let $T: R^3 \to R^2$ and $S: R^2 \to R^3$ be two Linear Trans. defined by
$T(x,y,z) = (x-3y-2z, y-4z)$ and
$S(x,y) = (2x, 4x-y, 2x+3y)$ find ST, TS, Is Product commute

Sol. Since Range of $S = R^3$ = domain of T

$\therefore$ TS is defined.

and $(TS)(x,y) = T[S(x,y)] = T[2x, 4x-y, 2x+3y]$

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Invertible operator:

A linear operator (Transformation) $T: V(F) \to V(F)$ is said to be invertible operator iff $\exists$ an operator $S: V(F) \to V(F)$ such that $TS = I = ST$, where I is an Identity operator. Here S is called the inverse of T and is denoted by T^{-1}

(Inverse of an invertible operator is unique).

1. Let $T: V_3(R) \to V_3(R)$ be defined as

(Not use this method) $T(x, y, z) = (3x, x-y, 2x+y+z)$ Prove that T is invertible and find T^{-1}!

Sol: We know that T is invertible iff T is one-one and onto

(i) To prove T is one-one:

Let $v_1 = (x_1, y_1, z_1)$ and $v_2 = (x_2, y_2, z_2) \in V_3(R)$

Such that $T(v_1) = T(v_2)$

ie $T(x_1, y_1, z_1) = T(x_2, y_2, z_2)$

$\Rightarrow (3x_1, x_1-y_1, 2x_1+y_1+z_1) = (3x_2, x_2-y_2, 2x_2+y_2+z_2)$

Comparing, we get

$x_1 = x_2$

$y_1 = y_2$

$z_1 = z_2$ ie $(x_1, y_1, z_1) = (x_2, y_2, z_2)$

$\Rightarrow v_1 = v_2$

$\therefore T(v_1) = T(v_2) \Rightarrow v_1 = v_2$

$\therefore T$ is one-one.

ii. To prove T is onto let $(a, b, c) \in V_3(R)$ and we shall show that $\exists$ a vector $(x, y, z) \in V_3(R)$ such that

$T(x, y, z) = (a, b, c)$

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$\Rightarrow (3x, x-y, 2x+y+z) = (a, b, c)$

$\Rightarrow \quad 3x = a, \quad x - y = b \quad 2x+y+z = c$

$x = a/3. \quad -y = b - x \quad$ and $\quad z = c - 2x - y$

$y = x - b \quad = c - 2\frac{a}{3} - (\frac{a}{3} - b)$

$= (a/3 - b) \quad = c - a + b$

$\therefore (x, y, z) = (\frac{a}{3}, \frac{a}{3} - b, c - a + b) \in V_3(R)$

Thus T is onto

Hence T is one-one and onto

$\Rightarrow$ T is invertible.

$\therefore T(x,y,z) = (a, b, c)$

$\Rightarrow T^{-1}(a,b,c) = (x, y, z)$

$= (\frac{a}{3}, \frac{a}{3} - b, c - a + b)$

$\therefore T^{-1}(a,b,c) = (\frac{a}{3}, \frac{a}{3} - b, c - a + b)$ is the required inverse of T.

(2). Let T be a linear operator on R^3 defined by

Not use This method

$T(x,y,z) = (2x, 4x-y, 2x+3y-z)$.

Show that T is invertible and find T^{-1}.

Sol.: We know that T is invertible iff $\exists$ a linear operator S on R^3 such that $ST = TS = I$

let $T(x,y,z) = (a, b, c)$

$\Rightarrow (2x, 4x-y, 2x+3y-z) = (a, b, c)$

$\therefore 2x = a \quad 4x - y = b \quad$ and $\quad 2x + 3y - z = c$

$x = \frac{a}{2} \quad 4x - b = y \quad z = 2x + 3y - c$

$= 2(\frac{a}{2}) + 3(2a - b) - c$

$y = 4\frac{a}{2} - b \quad = 7a - 3b - c$

$= 2a - b$

Define $S: R^3 \to R^3$ as

$$S(a,b,c) = \left(\frac{a}{2},\ 2a-b,\ 7a-3b-c\right)$$

(I) Check out the S is a linear operator.

II. $ST(x,y,z) = S\left[T(x,y,z)\right]$

$$= S\left[2x,\ 4x-y,\ 2x+3y-z\right]$$

$$= \left(\frac{2x}{2},\ 2(2x)-(4x-y),\ 7(2x)-3(4x-y) - (2x+3y-z)\right)$$

$$= (x,y,z).$$

$$= I(x,y,z)$$

and $TS(a,b,c) = T(S(a,b,c))$

$$= T\left[\frac{a}{2},\ 2a-b,\ 7a-3b-c\right]$$

$$= \left(2\left(\frac{a}{2}\right),\ 4\left(\frac{a}{2}\right) - (2a-b),\ 2\left(\frac{a}{2}\right)+3(2a-b) - (7a-3b-c)\right)$$

$$= (a,b,c) = I(a,b,c)$$

$\therefore\ ST = TS = I. \Rightarrow T$ is invertible and $T^{-1} = S$

ie $T^{-1}(a,b,c) = \left(\frac{a}{2},\ 2a-b,\ 7a-3b-c\right)$

(3) Let T be a linear operator on R^3 defined by

(use this method to find T^{-1}) $T(x,y,z) = (x-2y-z,\ y-z,\ x)$. Show that T is invertible, and find T^{-1}

Sol:- We know that T is invertible iff T is non-singular

To show that T is non-singular

let $T(x,y,z) = (0,0,0)$ for $(x,y,z) \in R^3$

$\Rightarrow (x-2y-z,\ y-z,\ x) = (0,0,0)$

$\Rightarrow x-2y-z=0,\quad y-z=0,\quad x=0$

$\therefore \boxed{x=0},\ \boxed{y=0},\ \boxed{z=0}$ ie $(x,y,z) = (0,0,0)$

$T(x,y,z) = (0,0,0) \Rightarrow (x,y,z) = (0,0,0)$

$\Rightarrow$ T is non-singular

$\Rightarrow$ T is investible operator on R^3.

To find T^{-1}

let $T(x,y,z) = (a,b,c)$

$\Rightarrow (x-2y-z,\ y-z,\ x) = (a,b,c)$

$\Rightarrow x-2y-z = a, \quad y-z=b, \quad x=c$

Solving $x=c$, $y = \dfrac{-a+b+c}{3}$, $z = \dfrac{-a-2b+c}{3}$,

Thus T^{-1} is given by $T^{-1}(a,b,c) = (x,y,z)$

$\Rightarrow T^{-1}(a,b,c) = \left(c, \dfrac{-a+b+c}{3}, -\dfrac{a-2b+c}{3}\right)$.

(4) Imp:

let $T: P_2(x) \to P_2(x)$ be linear operator defined by

$T(a+bx+cx^2) = (a+b)+(b+2c)x + (a+b+3c)x^2$

Show that T is investible and find T^{-1}.

Sol: we know that T is investible iff T is non-singular

To show T is non-singular:

let $T(a+bx+cx^2) = 0$ where $a+bx+cx^2 \in P_2x$ (0 is zero polynomial)

$\Rightarrow (a+b)+(b+2c)x+(a+b+3c)x^2 = 0+0x+0x^2$

$\Rightarrow a+b=0, \quad b+2c=0, \quad a+b+3c=0$

ie $a=0, \ b=0, \ c=0$ (on solving above three equations)

$\therefore a+bx+cx^2 = 0+0x+0x^2 = 0$

so that $T(a+bx+cx^2) = 0 \Rightarrow a+bx+cx^2 = 0$

$\Rightarrow T$ is non-singular $\Rightarrow T$ is investible

To find T^{-1}:

let $T(a+bx+cx^2) = \alpha+\beta x+\gamma x^2$

$\Rightarrow (a+b) + (b+2c)x + (a+b+3c)x^2 = \alpha+\beta x+\gamma x^2$

ie $a+b=\alpha, \quad b+2c=\beta, \quad a+b+3c=\gamma$

Solving, we get, $a = \dfrac{\alpha-3\beta+2\gamma}{3}$, $b = \dfrac{2\alpha+3\beta-2\gamma}{3}$, $c = \dfrac{\gamma-\alpha}{3}$

$\therefore T^{-1}$ is given by $T^{-1}(\alpha+\beta x+\gamma x^2) = a+bx+cx^2$

$\Rightarrow T^{-1}(\alpha+\beta x+\gamma x^2) = \dfrac{\alpha-3\beta+2\gamma}{3} + \dfrac{2\alpha+3\beta-2\gamma}{3}x + \dfrac{\gamma-\alpha}{3}x^2$.

Linear Transformations and Matrices

Def: Matrix representation of a linear transformation.

let $T: V \to W$ be a linear transformation, where V and W are vector space over a field F

and $\dim V = n$ and $\dim W = m$.

let $B_1 = \{v_1, v_2, --- v_n\}$

and $B_2 = \{w_1, w_2, --- w_m\}$ be ordered bases of V and W respectively

$\because$ $T: V \to W$ is a L.T (ie linear mapping) so that for every $v \in V$, we have $T(v) \in W$.

Since B_2 is a basis of W, so each $T(v) \in W$ can be uniquely written as a linear combination of the elements of B_2.

In particular, each $T(v_j) \in W$ where $1 \le j \le n$, can be expressed as follows:

$$T(v_1) = \alpha_{11} w_1 + \alpha_{21} w_2 + --- \alpha_{m1} w_m$$
$$T(v_2) = \alpha_{12} w_1 + \alpha_{22} w_2 + --- \alpha_{m2} w_m$$
$$- - - - - - - - -$$
$$- - - - - - - - -$$

ie $T(v_j) = \sum_{i=1}^{m} \alpha_{ij} w_i \quad 1 \le j \le n$

The coefficient matrix of the above equation is

$$\begin{bmatrix} \alpha_{11} & \alpha_{21} & \cdots & \alpha_{m1} \\ \alpha_{12} & \alpha_{22} & \cdots & \alpha_{m2} \\ - & - & - & - \\ \alpha_{1n} & \alpha_{2n} & \cdots & \alpha_{mn} \end{bmatrix}$$

The transpose of the above coefficient matrix is defined as the matrix of linear transformation T, relative to the bases and B_1 and B_2. (It is denoted by $[T; B_1, B_2]$ or simply by $[T]$.

Let T be a linear operator on R^2 defined by
$T(x,y) = (4x-2y,\ 2x+y)$
(i) find the matrix of T relative to the basis $B = \{(1,1); (-1,0)\}$
(ii) also verify that $[T;B][v;B] = [T(v);B]$ for any vector $v \in R^2$.

Sol:- firstly, we shall express any element

$v_1 = (\alpha, \beta) \in R^2$ as a linear combination of the elements

of basis B.

let $(\alpha, \beta) = a(1,1) + b(-1,0)$ for reals a and b

$\Rightarrow (\alpha, \beta) = (a-b, a)$

$\therefore \alpha = a-b, \quad \beta = a$

$\Rightarrow \alpha = \beta - b$ and $b = \beta - \alpha$

$\therefore (\alpha, \beta) = \beta(1,1) + (\beta - \alpha)(-1,0)$ ——— (1)

Given $T: R^2 \to R^2$ defined as

$T(x,y) = (4x-2y,\ 2x+y)$

and $B = \{(1,1), (-1,0)\}$ is a basis of R^2

Now $T(1,1) = (4-2,\ 2+1) = (2,3) = 3(1,1) + (3-2)(-1,0)$
$= 3(1,1) + 1(-1,0)$.

and $T(-1,0) = (-4-0,\ 2(-1)+0) = (-4,-2) = -2(1,1) + (-2+4)(-1,0)$
$= -2(1,1) + 2(-1,0)$

$$\therefore [T;B] = \begin{bmatrix} 3 & 1 \\ -2 & 2 \end{bmatrix}^{t} = \begin{bmatrix} 3 & -2 \\ 1 & 2 \end{bmatrix}$$

which is the matrix of T relative to the basis B.

To verify $[T;B][v;B] = [T(v);B]$

(54)

let $v = (x, y) \in R$

Then $v = (x, y) = y(1,1) + (y-x)(-1,0)$

$\therefore [v;B] = [y \quad y-x]^t = \begin{bmatrix} y \\ y-x \end{bmatrix}$

Now $T(v) = T(x,y) = (4x-2y, \; 2x+y)$

$= (2x+y)(1,1) + (2x+y-4x+2y)(-1,0)$

$= (2x+y)(1,1) + (-2x+3y)(-1,0)$

$\therefore [T(v);B] = [2x+y, \; -2x+3y]^t = \begin{bmatrix} 2x+y \\ -2x+3y \end{bmatrix}$

$L.H.S = [T;B][v;B] = \begin{bmatrix} 3 & -2 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} y \\ y-x \end{bmatrix}$

$= \begin{bmatrix} 3y-2(y-x) \\ y+2(y-x) \end{bmatrix}$

$= \begin{bmatrix} 2x+y \\ -2x+3y \end{bmatrix} = [T(v);B] = R.H.S$

Hence the result is verified.

(2) Let T be a linear operator on R^3 defined by

$T(x,y,z) = (2y+z, \; x-4y, \; 3x)$

(i) find the matrix of T relative to the basis

$B = \{(1,1,1), (1,1,0), (1,0,0)\}$

(ii) verify that $[T;B][v;B] = [T(v);B] \;\forall\; v \in R^3$.

(Not)

Sol:- (i) Firstly, we shall express any element $v_1 = (\alpha, \beta, \gamma) \in R^3$ as a linear combination of the elements of basis B

let $(\alpha, \beta, \gamma) = a(1,1,1) + b(1,1,0) + c(1,0,0)$ for some reals a, b, c

$(\alpha, \beta, \gamma) = (a+b+c, \; a+b, \; a)$

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...g these, we get

$$a = \gamma, \quad b = \beta - \gamma, \quad c = \alpha - \beta$$

$$\therefore (\alpha, \beta, \gamma) = \gamma(1,1,1) + (\beta - \gamma)(1,1,0) + (\alpha - \beta)(1,0,0) \quad \text{--- (1)}$$

Give $T: R^3 \to R^2$ is a linear operator defined as

$$T(x, y, z) = (2y+z,\ x-4y,\ 3x)$$

and $B = \{(1,1,1), (1,1,0), (1,0,0)\}$ is a basis of R^3

Now $T(1,1,1) = (2\times1+1,\ 1-4;\ 3\times1) = (3,-3,3)$

$$= 3(1,1,1) + (-3-3)(1,1,0) + (3+3)(1,0,0)$$

$$= 3(1,1,1) + (-6)(1,1,0) + 6(1,0,0)$$

$$T(1,1,0) = (2+0,\ 1-4,\ 3) = (2,-3,3)$$

$$= 3(1,1,1) + (-3-3)(1,1,0) + (2+3)(1,0,0)$$

$$= 3(1,1,1) + (-6)(1,1,0) + 5(1,0,0)$$

$$T(1,0,0) = (0+0,\ 1-0,\ 3) = (0,1,3)$$

$$= 3(1,1,1) + (1-3)(1,1,0) + (0-1)(1,0,0)$$

$$= 3(1,1,1) + (-2)(1,1,0) + (-1)(1,0,0)$$

$$\therefore [T;B] = \begin{bmatrix} 3 & -6 & 6 \\ 3 & -6 & 5 \\ 3 & -2 & -1 \end{bmatrix}' = \begin{bmatrix} 3 & 3 & 3 \\ -6 & -6 & -2 \\ 6 & 5 & -1 \end{bmatrix}$$

To verify that $[T;B][v;B] = [T(v);B] \quad \forall\, v \in R^3$

let $v = (x,y,z) \in R^3$

Then $v = (x,y,z) = z(1,1,1) + (y-z)(1,1,0) + (x-y)(1,0,0)$

$$\therefore [v;B] = [z \quad y-z \quad x-y]^t = \begin{bmatrix} z \\ y-z \\ x-y \end{bmatrix}$$

(56)

Now $T(v) = T(x, y, z)$

$= (2y+z, x-4y, 3x)$

$= 3x(1,1,1) + (x-4y-3x)(1,1,0) + (2y+z-x+4y)(1,0,0)$

$= 3x(1,1,1) + (-2x-4y)(1,1,0) + (-x+6y+z)(1,0,0)$

$= [3x, -2x-4y, -x+6y+z]^t$

$$= \begin{bmatrix} 3x \\ -2x-4y \\ -x+6y+z \end{bmatrix}$$

$$\text{L.H.S} = [T;B][v;B] = \begin{bmatrix} 3 & 3 & 3 \\ -6 & -6 & -2 \\ 6 & 5 & -1 \end{bmatrix} \begin{bmatrix} z \\ y-z \\ x-y \end{bmatrix}$$

$$= \begin{bmatrix} 3z + 3(y-z) + 3(x-y) \\ -6z - 6(y-z) - 2(x-y) \\ 6z + 5(y-z) - 1(x-y) \end{bmatrix}$$

$$= \begin{bmatrix} 3x \\ -2x-4y \\ -x+6y+z \end{bmatrix}$$

$= [T(v); B] = \text{R.H.S.}$

Hence the result is verified.

Type

Let $T: R^3 \to R^2$ be a linear transformation defined by
$T(x, y, z) = (3x+2y-4z, x-5y+3z)$

find the matrix of T in the following bases of R^3 and R^2:

$B_1 = \{(1,1,1), (1,1,0), (1,0,0)\}$

$B_2 = \{(1,3), (2,5)\}$

Sol: Given $T: R^3 \to R^2$ defined by

$T(x, y, z) = (3x+2y-4z, x-5y+3z)$

and $B_1 = \{(1,1,1), (1,1,0), (1,0,0)\}$,

$B_2 = \{(1,3), (2,5)\}$ be ordered basis for R^3 and R^2 respectively.

To find $[T: B_1, B_2]$

First of all we express any vector $v_1 = (\alpha, \beta) \in R^2$ as a linear combination of the element of basis B_2.

let $(\alpha, \beta) = a(1,3) + b(2,5)$

$= (a+2b, 3a+5b)$

$\Rightarrow a+2b = \alpha$

and $3a+5b = \beta$

$\Rightarrow a = -5\alpha+2\beta$ and $b = 3\alpha-\beta$

$\therefore (\alpha, \beta) = (-5\alpha+2\beta)(1,3) + (3\alpha-\beta)(2,5)$

Now $T(1,1,1) = (3+2-4, 1-5+3) = (1,-1)$

$= -7(1,3) + 4(2,5)$

$T(1,1,0) = (3+2-0, 1-5+0) = (5,-4) = -33(1,3) + 19(2,5)$

$T(1,0,0) = (3+0-0, 1-0+0) = (3,1) = (-13)(1,3) + 8(2,5)$

$$\therefore \ [T : B_1, B_2] = \begin{bmatrix} -7 & 4 \\ -33 & 19 \\ -13 & 8 \end{bmatrix}'$$

$$= \begin{bmatrix} -7 & -33 & -13 \\ 4 & 19 & 8 \end{bmatrix} \quad \text{Ans}$$

(4) let $T: R^3 \to R^2$ be the linear transformation defined by $T(x,y,z) = (2x+y-z, 3x-2y+4z)$.

find the matrix of T relative to ordered basis

$B_1 = \{(1,1,1), (1,1,0), (1,0,0)\}$

and $B_2 = \{(1,3), (1,4)\}$ of R^3 and R^2 respectively.

Ans $\begin{bmatrix} 3 & 4 & 5 \\ -1 & -8 & -3 \end{bmatrix}$

(5) find the matrix representation of the linear transformation $T: R^2 \to R^3$ defined by $T(x,y) = (3x-2y, 0, x+4y)$ with respect to ordered basis

$B_1 = \{(1,1), (0,2)\}$ and $B_2 = \{(1,1,0), (1,0,1), (0,1,1)\}$

for R^2 and R^3 respectively.

Ans - $\begin{bmatrix} -2 & -6 \\ 3 & 2 \\ 2 & 6 \end{bmatrix}$

ve?? Not need

Let $V(R)$ be the vector space of all 2×2 matrices and T be a linear operator on $V(R)$ such that $T(v) = Mv$ where $v \in V(R)$ and $M = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

find the matrix of T relative to basis

$$B = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$ of $V(F)$.

Sol: Given $T: V \to V$, a linear operator defined by

$$T(v) = Mv \text{ for all } v \in V(R) \text{ and } M = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

To find the matrix of T w.r.t the usual basis B of $V(R)$,

where $B = \{v_1, v_2, v_3, v_4\} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

Here $T(v_1) = T\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix}$

$T(v_2) = T\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 3 \end{bmatrix}$

$T(v_3) = T\left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 4 & 0 \end{bmatrix}$

$T(v_4) = T\left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 0 & 4 \end{bmatrix}$

Thus $T(v_1) = \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix} = 1\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + 0\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + 3\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + 0\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

$T(v_2) = \begin{bmatrix} 0 & 1 \\ 0 & 3 \end{bmatrix} = 0\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + 1\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + 0\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + 3\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

$T(v_3) = \begin{bmatrix} 2 & 0 \\ 4 & 0 \end{bmatrix} = 2\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + 0\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + 4\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + 0\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

$T(v_4) = \begin{bmatrix} 0 & 2 \\ 0 & 4 \end{bmatrix} = 0\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + 2\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + 0\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + 4\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

(58).

$$\therefore [T : B] = \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & 0 & 4 & 0 \\ 0 & 2 & 0 & 4 \end{bmatrix}' = \begin{bmatrix} 1 & 0 & 2 & \\ 0 & 1 & 0 & \\ 3 & 0 & 4 & \\ 0 & 3 & 0 & 4 \end{bmatrix}$$

<u>For Practice:</u>

<u>(4)</u>. find the matrix representation of $T: R^2 \to R^2$ defined as $T(x, y) = (3x - 4y, x + 5y)$ with respect to the basis

(i) $B = \{(1,0), (0,1)\}$ ———— Ans - $\begin{bmatrix} 3 & -4 \\ 1 & 5 \end{bmatrix}$

(2) $B = \{(1,3), (3,4)\}$ ———— Ans $\begin{bmatrix} 84/5 & 97/5 \\ -43/5 & -44/5 \end{bmatrix}$

<u>(5)</u> find the matrix representation of each of following linear operators relative to given basis of R^3

<u>(a)</u> $T: R^3 \to R^3$ is defined by

$$T(x, y, z) = (2z, x - 2y, x + 2y)$$

and basis $B_1 = \{(1,2,1), (1,1,1), (1,1,0)\}$

<u>Ans</u> $\begin{bmatrix} -5 & -3 & -1 \\ 10 & 6 & 4 \\ -3 & -1 & -3 \end{bmatrix}$

<u>(b)</u> $T: R^3 \to R^3$ is defined by

$$T(x, y, z) = (3x + z, -2x + y, -x + 2y + 4z)$$

and basis is $B_2 = \{(1,0,1), (-1,2,1), (2,1,1)\}$

$$\begin{bmatrix} \frac{17}{4} & \frac{35}{4} & \frac{22}{4} \\ -3/4 & \frac{15}{4} & -6/4 \\ -\frac{1}{2} & -\frac{7}{2} & 0 \end{bmatrix}$$

<u>Ans</u>.

Mathematics